**Analytical RVE theory of dispersed media and its applications**

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1. Introduction

This note introduces an approach inspired by structural sums, which describes the interactions among inclusions of dispersed heterogeneous media. The structural sums serve as the cornerstone for mathematical models of random structures within the framework of the *analytical* Representative Volume Element (*a*RVE) theory. We quantified the inter-phase interactions within media and derived analytical formulas for their macroscopic properties. Application to the diagnosis and classification of glioma structures requires a set of corresponding pictures to develop the corresponding machine learning models, and can be performed together with the specialists in medicine.

1. Description of the problem

Gliomas, a type of brain and spinal cord tumor, are classified based on their cellular origin and molecular characteristics. The classification incorporates molecular data alongside traditional histological features and is based on simple observations and expert visual evaluations. This evolving understanding is necessary for the integration of molecular profiles into the diagnosis and classification of gliomas. Convolutional neural networks were introduced for automated, multiclass classification of glioma grades in [1]. This straightforward usage is based on the machine learning approach. It is supposed that the introduction of images and their corresponding deep processing by the almighty computer will lead to a result. The main difficulty for a wide application of the above approach is the huge number of features. On the other side, the fundamental characteristics of cellular dynamics can be locally approximated by diffusion, fluid transport, thermal conduction, and other chemo-physical processes within multiphase media. These phenomena are represented by systems of ordinary and partial differential equations from classical mathematical physics, such as the Laplace, Poisson, Navier–Stokes equations, and their coupled variants [2]. While this modeling framework yields insightful and generalizable results, its practical implementation typically relies on purely numerical solvers, which impose limitations due to high computational complexity and resource demands. The problem of essential reduction of the principal parameters becomes decisive in the considered problem of brain and spinal cord tumors classification.

Related work

Similar classification problems concerning dispersed composites have been studied for many years, beginning with Maxwell’s and Rayleigh's works. Various approaches based on engineering observations, empirical models, and finite method simulations are very popular in material sciences. However, some analytical formulas were given without an analysis of their precision. This led, at most, to formally different but asymptotically equivalent formulas for the same classes of composites. Typical methods and tricks used by self-consistent methods and their modifications contradict the principles of homogenization and asymptotic models discussed in [3, Chapter 9] and [4]. Randomness in composites is strictly related to the measure theory. In the considered case of non-overlapping inclusions, the measure theory is applied to the characteristic set equal to a tensor 𝜀1 in inclusions and to 𝜀2 in the matrix. Theoretically, any such measurable set is determined by the infinite set of 𝑛-point correlation functions. Implementing multiple correlation functions for random composites is usually reduced to the well-studied spatial two-point correlation functions. Theoretically, the infinite set of multiple correlation functions completely describes a random composite. However, the virtual impossibility of computing the correlation functions of higher orders restricts their applications. A computationally effective method of structural sums was proposed in [5] and works cited therein. This method is based on the generalized Schwarz alternating method and the solution to the Riemann-Hilbert and ℝ-linear problems for an arbitrary multiply connected domain.

1. Solution to the problem

The approach described in [5] and works cited therein leads to the theory of *a*RVE. It can be considered a constructive application of the Decomposition Theorem [5]. One of the *a*RVE applications is outlined below. Let us have at our disposal two plane pictures of dispersed composites and their digital treatment in the form of two sets of the centers of inclusions, **a** = {*𝑎*1*, 𝑎*2*, . . . , 𝑎*N} and **a**′ = {*𝑎’*1*, 𝑎’*2*, . . . , 𝑎’*N}, scaled to the periodic unit square cell. We want to know whether these two media belong to the same class of random composites. First, we must calculate the structural sums described in [5]for two sets of points, **a** and **a**′, expressed by the vector sets **E** ={*e*1*, e*2*, . . .* } and **E**′ = {*e’*1*, e’*2*, . . .* }, and then compare them. The Decomposition Theorem [5] guarantees the complete geometric characterization of macroscopic properties of media represented by sets **a** and **a**′. The question of macroscopic anisotropy can be resolved by the same method.

1. Conclusions and future work

The main advantage of *a*RVE compared to other methods can be summarized as follows:
• A class of random composites can be directly determined by a set of structural sums without the computation of its effective properties.
• The number of inclusions per periodicity cell is practically not restricted.
• The method doesn’t use a virtually impossible computation of higher order
correlation functions.

The derived theoretical model can be applied to the diagnosis and classification of gliomas. It requires a set of corresponding pictures to develop the corresponding machine learning models, and can be performed together with the specialists in medicine.

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